1 Spaces and subspaces

Vector Space Definition The set $\mathcal{V}$ is called a *vector space over* $\mathbb{F}$ when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{V}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. This is called the *closure property for vector addition.*

(A2) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.

(A3) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\boldsymbol{x}+\mathbf{0} = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathcal{V}$.

(A4) For each $\boldsymbol{x} \in \mathcal{V}$, there is an element $(-\boldsymbol{x}) \in \mathcal{V}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.

(A5) $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$ for every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

(M1) $\alpha\boldsymbol{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{F}$ and $\boldsymbol{x} \in \mathcal{V}$. This is the *closure property for scalar multiplication.*

(M2) $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x})$ for all $\alpha, \beta \in \mathbb{F}$ and every $\boldsymbol{x} \in \mathcal{V}$.

(M3) $\alpha(\boldsymbol{x}+\boldsymbol{y}) = \alpha\boldsymbol{x} + \beta\boldsymbol{y}$ for every $\alpha \in \mathbb{F}$ and all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\boldsymbol{x} = \alpha\boldsymbol{x}+\beta\boldsymbol{x}$ for all $\alpha, \beta \in \mathbb{F}$ and every $\boldsymbol{x} \in \mathcal{V}$.

(M5) $1\boldsymbol{x} = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathcal{V}$.

1. Let $\mathcal{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$ be a given set, and define the operations addition $+$ and scalar multiplication $\cdot$ on the following way

$$\text{addition } + : \quad \forall (x, y), (a, b) \in \mathcal{V}$$

$$(x, y) + (a, b) = (x + a, y + b),$$

$$\text{scalar multiplication:} \quad \forall \alpha \in \mathbb{R}, \quad \forall (x, y) \in \mathcal{V}$$

$$\alpha(x, y) = (\alpha y, \alpha x).$$

Is the set $\mathcal{V}$ a vector space over $\mathbb{R}$? Explain your answer!

2. Show that the set $\mathcal{V} = \{(x, x, y) \mid x, y \in \mathbb{R}\}$ is a vector space if the vector addition and scalar multiplication operations are defined on the following way

$$\text{vector addition: } \forall (x, x, y), (a, a, b) \in \mathcal{V}$$

$$(x, x, y) + (a, a, b) = (x + a, x + a, y + b),$$

$$\text{scalar multiplication: } \forall \alpha \in \mathbb{R}, \quad \forall (x, x, y) \in \mathcal{V}$$

$$\alpha(x, x, y) = (\alpha x, \alpha x, \alpha y).$$

3. Why must a real or complex nonzero vector space contain an infinite number of vectors?

4. Are the following sets with a given operations vector spaces? Explain your answer! (a) Set $\mathbb{R}_0^+$, all no-negative real numbers, with usual addition and scalar multiplication. (b) Set $\mathcal{V}$ of all polynomials or order ≥ 3, including 0; operations are standard addition of polynomials, and standard scalar multiplication. (c) Set $\mathcal{V}$ of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, with operations from $\mathrm{Mat}_{2\times2}(\mathbb{R})$. (d) Set $\mathcal{V}$ of all 2×2 matrices with equal sum of entries in each column; operations from $\mathrm{Mat}_{2\times2}(\mathbb{R})$. (e) Set $\mathcal{V}$ of all 2×2 matrices which determinant is equal to zero; operations from $\mathrm{Mat}_{2\times2}(\mathbb{R})$. (g) Set $\mathcal{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$ with ordinary addition, but scalar multiplication $\alpha(x, y) = (x, y)$ for all $\alpha \in \mathbb{R}$.

Subspaces Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathbb{F}$ (symbolically, $\mathcal{S} \subseteq \mathcal{V}$). If $\mathcal{S}$ is also a vector space over $\mathbb{F}$ using the same addition and scalar multiplication operations, then $\mathcal{S}$ is said to be a *subspace* of $\mathcal{V}$. It's not necessary to check all 10 of the defining conditions in order to determine if a subset is also a subspace - only the closure conditions (A1) and (M1) need to be considered. That is, a nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

(A1) $\boldsymbol{x},\ \boldsymbol{y} \in \mathcal{S} \Longrightarrow \boldsymbol{x}+\boldsymbol{y} \in \mathcal{S}$ and (M1) $\boldsymbol{x} \in \mathcal{S} \Longrightarrow \alpha\boldsymbol{x} \in \mathcal{S}$ for all $\alpha \in \mathbb{F}$.

5. (i) Show that the set $\mathcal{V} = \{(x, -x) \mid x \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^2$. (ii) For a given vector space $\mathcal{V}$, let $\mathcal{U} = \{\mathbf{0}\}$ be a set containing only the zero vector. Show that $\mathcal{U}$ is a vector subspace of $\mathcal{V}$.

6. Determine which of the following subsets of $\mathbb{R}^n$ are in fact subspaces of $\mathbb{R}^n$ $(n > 2)$. (a) $\{\boldsymbol{x} \mid x_i \geq 0\}$, (b) $\{\boldsymbol{x} \mid x_1 = 0\}$, (c) $\{\boldsymbol{x} \mid x_1 x_2 = 0\}$, (d) $\{\boldsymbol{x} \mid \sum_{j=1}^n x_j = 0\}$, (e) $\{\boldsymbol{x} \mid \sum_{j=1}^n x_j = 1\}$, (f) $\{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},$ where $A_{m\times n} \neq \mathbf{0}$ and $\boldsymbol{b}_{m\times 1} \neq 0\}$.

7. Determine which of the following subsets of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ are in fact subspaces of $\mathrm{Mat}_{n\times n}(\mathbb{R})$. (a) The symmetric matrices. (b) The diagonal matrices. (c) The nonsingular matrices. (d) The singular matrices. (e) The triangular matrices. (f) The upper-triangular matrices. (g) All matrices that commute with a given matrix A. (h) All matrices such that $A^2 = A$. (i) All matrices such that $\mathrm{trace}(A) = 0$.

Spanning Sets

- For a set of vectors $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$, the subspace $\mathrm{span}(\mathcal{S}) = \{\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + ... + \alpha_r\boldsymbol{v}_r : \alpha_i \in \mathbb{F}\}$ generated by forming all linear combinations of vectors from $\mathcal{S}$ is called the *space spanned by* $\mathcal{S}$.
- If $\mathcal{V}$ is a vector space such that $\mathcal{V} = span(\mathcal{S})$, we say $\mathcal{S}$ is a *spanning set* for $\mathcal{V}$. In other words, $\mathcal{S}$ *spans* $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

8. In the following it is given a set $\mathcal{S}$. Describe what is $\mathrm{span}(\mathcal{S})$ and if it is possible give geometrical picture. (i) Let $u, v \in \mathbb{R}^3$ denote two noncollinear vectors, and let $\mathcal{S} = \{u, v\}$. (ii) $\mathcal{S} = \{(1,1)^\top, (2,2)^\top\}$. (iii) $\mathcal{S}$ contains unit vectors $\{e_1 = (1,0,0)^\top, e_2 = (0,1,0)^\top, e_3 = (0,0,1)^\top\}$. (iv) $\mathcal{S} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$ is set of unit vectors from $\mathbb{R}^n$. (v) $\mathcal{S} = \{1, x, x^2, ..., x^n\}$ (vi) $\mathcal{S} = \{1, x, x^2, ...\}$.

9. Carefully explain is it true that $\mathrm{span}(\mathcal{S}) = \mathbb{R}^3$, if we have that $\mathcal{S} = \{(1,1,1), (1,-1,-1), (3,1,1)\}$.

10. Which of the following are spanning sets for $\mathbb{R}^3$? (a) $\{(1,1,1)\}$ (b) $\{(1,0,0), (0,0,1)\}$, (c) $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$, (d) $\{(1,2,1), (2,0,-1), (4,4,1)\}$, (e) $\{(1,2,1), (2,0,-1), (4,4,0)\}$.

11. For a set of vectors $\mathcal{S} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_n\}$ from a subspace $\mathcal{V} \subseteq \mathrm{Mat}_{m\times 1}(\mathbb{R})$, let A be the matrix containing the $\boldsymbol{a}_i$'s as its columns. Explain why $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\boldsymbol{b} \in \mathcal{V}$ there corresponds a column $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ (i.e., if and only if $A\boldsymbol{x} = \boldsymbol{b}$ is a consistent system for every $\boldsymbol{b} \in \mathcal{V}$).

Sum of Subspaces If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$ then the sum of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the set of all possible sums of vectors from $\mathcal{X}$ with vectors from $\mathcal{Y}$. That is,

$$\mathcal{X} + \mathcal{Y} = \{\boldsymbol{x} + \boldsymbol{y} \,|\, \boldsymbol{x} \in \mathcal{X} \text{ and } \boldsymbol{y} \in \mathcal{Y}\}.$$

- The sum $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$.
- If $\mathcal{S}_\mathcal{X}$, $\mathcal{S}_\mathcal{Y}$ span $\mathcal{X}$, $\mathcal{Y}$ then $\mathcal{S}_\mathcal{X} \cup \mathcal{S}_\mathcal{Y}$ spans $\mathcal{X} + \mathcal{Y}$.

12. If $\mathcal{X}$ is a plane passing through the origin in $\mathbb{R}^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

13. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq V$, explain why $\mathrm{span}(\mathcal{M} \cup \mathcal{N}) = \mathrm{span}(\mathcal{M}) + \mathrm{span}(\mathcal{N})$.

14. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$. (a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$. (b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

15. For $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $\mathcal{S} \subseteq \mathrm{Mat}_{n\times 1}(\mathbb{R})$, the set $A(\mathcal{S}) = \{Ax \mid x \in \mathcal{S}\}$ contains all possible products of A with vectors from $\mathcal{S}$. We refer to $A(\mathcal{S})$ as the set of *images* of $\mathcal{S}$ under A. (a) If $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, prove $A(\mathcal{S})$ is a subspace of $\mathbb{R}^m$. (b) If $\boldsymbol{s}_1, \boldsymbol{s}_2, ..., \boldsymbol{s}_k$ spans $\mathcal{S}$, show $A\boldsymbol{s}_1, A\boldsymbol{s}_2, ..., A\boldsymbol{s}_k$ spans $A(\mathcal{S})$.

16. Let $\mathcal{L} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0\right\}$. Show that $\mathcal{L}$ is subspace of vector space $\mathbb{R}^3$.

17. Let $\mathcal{V} = \mathbb{R}^n$ and let $(a_1, a_2, ..., a_n)^\top$ be some fixed vector from $\mathcal{V}$. Show that the family of all elements $(x_1, x_2, ..., x_n)^\top$ from $\mathcal{V}$ with the property $a_1x_1 + ... + a_nx_n = 0$ is subspace of vector space $\mathcal{V}$. In another words, show that

$$\mathcal{M} = \{(x_1, x_2, ..., x_n)^\top \in \mathcal{V} \,|\, a_1x_1 + ... + a_nx_n = 0\}$$

is subspace of $\mathcal{V}$.

18. Let $\mathcal{V}$ denote vector space of all matrices of form 2×2 over the field of real numbers. Let $\mathcal{W}_1$ be the set of all matrices of form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and let $\mathcal{W}_2$ be the set of all matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

Show that $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathcal{V}$.

19. Let

$$\mathcal{V} = \left\{ \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,\middle|\, z_1 - 2\overline{z_2} + z_3 = 0, z_1 + \overline{z_2 + z_3} + z_4 = 0 \right\}$$

be a given set. Show that $\mathcal{V}$ is real subspace of vector space $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

InC: 1, 3, 6, 8, 9, 12, 15. HW: 16, 17, 18, 19.